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SENSITIVITY THEOREMS IN INTEGER LINEAR PROGRAMMING

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We consider integer linear programming problems with a fixed coefficient matrix and varying objective function and right-hand-side vector. Among our results, we show that, for any optimal solution to a linear program $\max\{wx\colon Ax \leq b\}$, the distance to the nearest optimal solution to the corresponding integer program is at most the dimension of the problem multiplied by the largest subdeterminant of the integral matrix A. Using this, we strengthen several integer programming 'proximity' results of Blair and Jeroslow; Graver; and Wolsey. We also show that the Chvátal rank of a polyhedron $\{x\colon Ax \leq b\}$ can be bounded above by a function of the matrix A, independent of the vector b, a result which, as Blair observed, is equivalent to Blair and Jeroslow's theorem that 'each integer programming value function is a Gomory function.'

Key words: Integer Linear Programming, Chvátal Rank, Cutting Planes, Sensitivity Analysis.

1. Introduction

For a given integer program $\max\{wx\colon Ax \leq b, x \text{ integral}\}$, how does the set of optimal solutions change as the vectors w and b are varied? Early work on this topic was carried out by Gomory [14, 15], who considered the connection between optimal solutions to an integer program and its linear programming relaxation for a range of right-hand-side vectors b. His work was continued and extended by Wolsey [28]. Other studies have been made by Blair and Jeroslow [1, 2, 3].

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In this paper, we consider several different aspects of the problem. We first show (Theorem 1) that, for any optimal solution to a linear program $\max\{wx: Ax \leq b\}$, where A is an integral matrix, the distance to the nearest optimal solution to the corresponding integer program is at most the number of variables multiplied by the largest subdeterminant of A. This implies results of Blair and Jeroslow [2], von zur Gathen and Sieveking [10], and Wolsey [28]. Next (Theorem 5), we sharpen a result of Blair and Jeroslow [1], which implies that a change in the right-hand-side vector cannot produce more than an affine change in the optimal value of an integer program. We then show (Theorem 6) that for any nonoptimal integral solution to $\max\{wx: Ax \leq b, x \text{ integral}\}$ there exists an integral solution nearby (for a fixed matrix A) which has a greater objective value. This improves a result of Graver [16] and Blair and Jeroslow [3]. Finally, we show (Theorem 10) that the Chvátal rank of a polyhedron $\{x: Ax \leq b\}$ can be bounded above by a function of the matrix A, independent of the vector b.

We assume throughout the paper that all polyhedra, matrices, and vectors are rational. The l_∞-norm of a vector $x = (x_1, \ldots, x_n)$ is denoted by $\|x\|_\infty = \max\{|x_i|: i = 1, \ldots, n\}$ and the l_1-norm is denoted by $\|x\|_1 = \sum\{|x_i|: i = 1, \ldots, n\}$. For a matrix A, we denote by $\Delta(A)$ the maximum of the absolute values of the determinants of the square submatrices of A. If A is an $m \times n$ matrix and b is an m-component vector, then $(A|b)$ denotes the matrix obtained by adjoining b, as an $n+1$st column, to A. The greatest integer less than or equal to a number β is denoted by $\lfloor \beta \rfloor$.

For basic results in the theory of polyhedra and integer linear programming, the reader is referred to Schrijver [26] and Stoer and Witzgall [27].

2. Proximity results

We begin with a theorem on the distance between optimal solutions to an integer program and its linear programming relaxation.

Theorem 1. *Let A be an integral $m \times n$ matrix and let b and w be vectors such that $Ax \leq b$ has an integral solution and $\max\{wx: Ax \leq b\}$ exists. Then*

(i) *for each optimal solution $\bar{x}$ to $\max\{wx: Ax \leq b\}$ there exists an optimal solution z^* to $\max\{wx: Ax \leq b, x \text{ integral}\}$ with $\|\bar{x} - z^*\|_\infty \leq n\Delta(A)$*

and

(ii) *for each optimal solution $\bar{z}$ to $\max\{wx: Ax \leq b, x \text{ integral}\}$ there exists an optimal solution x^* to $\max\{wx: Ax \leq b\}$ with $\|\bar{z} - x^*\|_\infty \leq n\Delta(A)$.*

Proof. Let $\bar{x}$ and $\bar{z}$ be optimal solutions to $\max\{wx: Ax \leq b\}$ and $\max\{wx: Ax \leq b, x \text{ integral}\}$ respectively. Split A into submatrices A_1, A_2 such that $A_1\bar{x} \geq A_1\bar{z}$ and $A_2\bar{x} < A_2\bar{z}$. Since $a_i\bar{x} < b_i$ for each inequality $a_i x \leq b_i$, of $Ax \leq b$, with a_i a row of A_2, the dual variables corresponding to the rows of A_2 are equal to zero in every optimal solution to the dual linear program of $\max\{wx: Ax \leqslant b\}$. So there exists a vector $y^1 \geqslant 0$ such that $y^1 A_1 = w$. This implies that $w\tilde{x} \geqslant 0$ for each vector $\tilde{x}$ in the cone $C = \{x: A_1 x \geqslant 0, A_2 x \leqslant 0\}$. Let G be a finite set of integral vectors which generates C (so C is the set of vectors which can be written as a nonnegative linear combination of vectors in G). Using Cramer's rule, we may assume that $\|g\|_\infty \leqslant \Delta(A)$ for each $g \in G$. As $\bar{x} - \bar{z} \in C$, there exists, by Carathéodory's theorem, a set $\{g^1, \ldots, g^t\} \subseteq G$ and numbers $\lambda_i \geqslant 0$, $i = 1, \ldots, t$, such that $\bar{x} - \bar{z} = \lambda_1 g^1 + \cdots + \lambda_t g^t$, where t is the dimension of C.

To verify (i), let

$$z^* = \bar{z} + \lfloor \lambda_1 \rfloor g^1 + \cdots + \lfloor \lambda_t \rfloor g^t = \bar{x} - (\lambda_1 - \lfloor \lambda_1 \rfloor) g^1 - \cdots - (\lambda_t - \lfloor \lambda_t \rfloor) g^t. \quad (1)$$

Since $\bar{z}$ is integral and $g^1, \ldots, g^t$ are integral, z^* is also integral. Furthermore,

$$A_2 z^* = A_2 \bar{z} + \lfloor \lambda_1 \rfloor A_2 g^1 + \cdots + \lfloor \lambda_t \rfloor A_2 g^t \leqslant A_2 \bar{z} \quad \text{and}$$

$$A_1 z^* = A_1 \bar{x} - (\lambda_1 - \lfloor \lambda_1 \rfloor) A_1 g^1 - \cdots - (\lambda_t - \lfloor \lambda_t \rfloor) A_1 g^t \leqslant A_1 \bar{x}.$$

So $Az^* \leqslant b$. Now since $wg^i \geqslant 0$ for $i = 1, \ldots, t$, we have that $wz^* \geqslant w\bar{z}$ and, hence, that z^* is an optimal solution to $\max\{wx: Ax \leqslant b, x \text{ integral}\}$. (This implies that $wg^i = 0$ for all $i \in \{1, \ldots, t\}$ with $\lambda_i \geqslant 1$, a fact which is used below.) Finally,

$$\|\bar{x} - z^*\|_\infty = \|(\lambda_1 - \lfloor \lambda_1 \rfloor) g^1 + \cdots + (\lambda_t - \lfloor \lambda_t \rfloor) g^t\|_\infty \leqslant \|g^1\|_\infty + \cdots + \|g^t\|_\infty. \quad (2)$$

So $\|\bar{x} - z^*\|_\infty \leqslant t\Delta(A) \leqslant n\Delta(A)$.

To verify (ii), let

$$x^* = \bar{x} - \lfloor \lambda_1 \rfloor g^1 - \cdots - \lfloor \lambda_t \rfloor g^t = \bar{z} + (\lambda_1 - \lfloor \lambda_1 \rfloor) g^1 + \cdots + (\lambda_t - \lfloor \lambda_t \rfloor) g^t. \quad (3)$$

Using the above arguments, it follows that $Ax^* \leqslant b$ and $\|\bar{z} - x^*\|_\infty \leqslant n\Delta(A)$. Since $wg^i = 0$ for all $i \in \{1, \ldots, t\}$ with $\lambda_i \geqslant 1$, we have $wx^* = w\bar{x}$. So x^* is an optimal solution to $\max\{wx: Ax \leqslant b\}$. □

This result strengthens a theorem of Blair and Jeroslow [2, Theorem 1.2], who showed that for any fixed matrix A and fixed vector w, there exists a constant T such that for any optimal solution $\bar{x}$ to $\max\{wx: Ax \leqslant b\}$, there exists an optimal solution $\bar{z}$ to $\max\{wx: Ax \leqslant b, x \text{ integral}\}$ such that $\|\bar{x} - \bar{z}\|_\infty \leqslant T$ (assuming that $Ax \leqslant b$ has an integral solution). (In fact, an analysis of Blair and Jeroslow's proof (see also the proof of [1, Theorem 2.1(1)]) will show that T is independent of w.) Similarly, the following consequence of Theorem 1 strengthens a result of Blair and Jeroslow [1, Theorem 2.1(2), 3, Corollary 4.7] on the difference between the optimal value of an integer program and its linear programming relaxation.

Corollary 2. *Let A be an integral $m \times n$ matrix and let b and w be vectors such that $Ax \leqslant b$ has an integral solution and $\max\{wx: Ax \leqslant b\}$ exists. Then*

$$\max\{wx: Ax \leqslant b\} - \max\{wx: Ax \leqslant b, x \textit{ integral}\} \leqslant n\Delta(A)\|w\|_1. \quad \square \quad (4)$$

Another consequence of Theorem 1 is the following result of von zur Gathen and Sieveking [10] (which implies that integer programming (feasibility) is in the class NP).

Corollary 3. *Let A be an integral $m \times n$ matrix and b an integral m-component vector. Then if $Ax \leqslant b$ has an integral solution, then it has one with components at most $(n+1)\Delta((A|b))$ in absolute value.*

Proof. Suppose that $Ax \leqslant b$ has an integral solution. There exists a vector $\bar{x}$ with $A\bar{x} \leqslant b$ such that the nonzero components of $\bar{x}$ are given by $B^{-1}\tilde{b}$ for some submatrix B of A and some part $\tilde{b}$ of b. We have $\|\bar{x}\|_\infty \leqslant \Delta((A|b))$. By Theorem 1, there exists an integral vector $\bar{z}$ such that $A\bar{z} \leqslant b$ and $\|\bar{z}-\bar{x}\|_\infty \leqslant n\Delta(A)$. So

$$\|\bar{z}\|_\infty \leqslant \|\bar{z}-\bar{x}\|_\infty + \|\bar{x}\|_\infty \leqslant n\Delta(A) + \Delta((A|b)) \leqslant (n+1)\Delta((A|b)). \quad \square \qquad (5)$$

A third consequence of Theorem 1 is a result of Wolsey [28], which shows that an integer program can be solved by first solving its linear programming relaxation and then checking a finite set of lower dimensional 'correction vectors'. Wolsey [28] works with linear programs of the form $\max\{wx: Ax = b, x \geqslant 0\}$, where A is an $m \times n$ matrix of rank m. If such a linear program has an optimal solution, then it has one of the form $\bar{x}_B = B^{-1}b$, $\bar{x}_N = 0$ where B is a basis of A (that is, a $m \times m$ nonsingular submatrix of A), x_B are those variables corresponding to columns of B and x_N are those variables corresponding to columns of N (the submatrix of A formed by those columns not in B). Such a basis B is an *optimal basis.*

Corollary 4. *Let A be an integral $m \times n$ matrix of rank m. Then there exists a finite set V of nonnegative, integral $(n-m)$-component vectors such that: For any vectors b and w for which* $\max\{wx: Ax = b, x \geqslant 0\}$ *has an optimal solution and any optimal basis B, if $Ax = b$, $x \geqslant 0$ has an integral solution then for some vector $v \in V$ an optimal solution to* $\max\{wx: Ax = b, x \geqslant 0, x \text{ integral}\}$ *is $\bar{x}_N = v$, $\bar{x}_B = B^{-1}b - B^{-1}Nv$.*

Proof. Let $V = \{v \in Z^{n-m}: \|v\|_\infty \leqslant n\Delta(A), v \geqslant 0\}$. Suppose that B is an optimal basis for $\max\{wx: Ax = b, x \geqslant 0\}$ and let $\bar{x}_B = B^{-1}b$, $\bar{x}_N = 0$ be the corresponding optimal solution. By Theorem 1, there exists an optimal solution $\bar{z}$ to $\max\{wx: Ax = b, x \geqslant 0, x \text{ integral}\}$ with $\|\bar{x}-\bar{z}\|_\infty \leqslant n\Delta(A)$. Thus $\|\bar{z}_N\|_\infty \leqslant n\Delta(A)$, which implies that $\bar{z}_N \in V$. $\square$

For a fixed matrix A, Theorem 1 shows that optimal solutions to an integer program $\max\{wx: Ax \leqslant b, x \text{ integral}\}$ and its linear programming relaxation are near to each other. Our next theorem, which is a sharpened form of the integer programming 'strong proximity result' of Blair and Jeroslow [1], shows that for small changes in the right-hand-side vector b, optimal solutions to the corresponding integer programs are near to each other. Assertion (i) extends results of Hoffman [19] and Mangasarian [23].

Theorem 5. *Let A be an integral $m\times n$ matrix and let b, b', and w be vectors such that* $\max\{wx\colon Ax\leq b\}$ *and* $\max\{wx\colon Ax\leq b'\}$ *each have optimal solutions. Then*

(i) *for each optimal solution $\bar{x}$ to* $\max\{wx\colon Ax\leq b\}$ *there exists an optimal solution $\bar{x}'$ to* $\max\{wx\colon Ax\leq b'\}$ *with $\|\bar{x}-\bar{x}'\|_\infty\leq n\Delta(A)\|b-b'\|_\infty$, and*

(ii) *if $Ax\leq b$ and $Ax\leq b'$ each have integral solutions, then for each optimal solution $\bar{z}$ to* $\max\{wx\colon Ax\leq b, x \text{ integral}\}$ *there exists an optimal solution $\bar{z}'$ to* $\max\{wx\colon Ax\leq b', x \text{ integral}\}$ *with $\|\bar{z}-\bar{z}'\|_\infty\leq n\Delta(A)(\|b-b'\|_\infty+2)$.*

Proof. We first show part (i) in the case where w is the zero vector, that is, if $\bar{x}$ is a solution of $Ax\leq b$ then $A\bar{x}'\leq b'$ for some $\bar{x}'$ with $\|\bar{x}-\bar{x}'\|_\infty\leq n\Delta(A)\|b-b'\|_\infty$. Suppose such an $\bar{x}'$ does not exist. Then the system

$$\begin{aligned} &Ax\leq b',\\ &x\leq \bar{x}+\varepsilon\mathbf{1}, \\ &-x\leq -\bar{x}+\varepsilon\mathbf{1} \end{aligned} \tag{6}$$

(where $\varepsilon=n\Delta(A)\|b-b'\|_\infty$ and $\mathbf{1}=(1,\dots,1)^{\mathrm{T}}$) has no solution. By Farkas' Lemma, we have

$$yA+u-v=0,\qquad yb'+u(\bar{x}+\varepsilon 1)+v(-\bar{x}+\varepsilon 1)<0 \tag{7}$$

for some nonnegative vectors y, u, v. As $Ax\leq b'$ has a solution, we have $u+v\neq 0$. We may assume $\|u+v\|_1=1$. We may also assume, by Carathéodory's theorem, that the positive components of y correspond to linearly independent rows of A. So the positive part of y is equal to $B^{-1}(-\tilde{u}+\tilde{v})$ for some parts $\tilde{u}$, $\tilde{v}$ of u, v and some submatrix B of A. Hence, $\|y\|_1\leq n\Delta(A)\|u-v\|_1\leq n\Delta(A)\|u+v\|_1=n\Delta(A)$. Now we have the contradiction

$$\begin{aligned} 0&>yb'+u(\bar{x}+\varepsilon 1)+v(-\bar{x}+\varepsilon 1)=yb'-yA\bar{x}+\varepsilon\|u+v\|_1\\ &\geq y(b'-b)+\varepsilon\geq -\|y\|_1\|b-b'\|_\infty+\varepsilon\geq 0. \end{aligned} \tag{8}$$

We next show part (i) in general. Let $\bar{x}$ be an optimal solution to $\max\{wx\colon Ax\leq b\}$ and let x^* be any optimal solution to $\max\{wx\colon Ax\leq b'\}$. Let $A_0x\leq b_0$ be those inequalities from $Ax\leq b$ that are satisfied by $\bar{x}$ with equality. Then $yA_0=w$ for some $y\geq 0$ (by the duality theorem of linear programming). Since $\bar{x}$ satisfies:

$$A\bar{x}\leq b,\qquad A_0\bar{x}\geq A_0x^*-\|b-b'\|_\infty 1 \tag{9}$$

and since x^* is a solution of $[Ax\leq b', A_0x\geq A_0x^*]$, we have from above that $[A\bar{x}'\leq b', A_0\bar{x}'\geq A_0x^*]$ for some $\bar{x}'$ with $\|\bar{x}-\bar{x}'\|_\infty\leq n\Delta(A)\|b-b'\|_\infty$. As $w\bar{x}'=yA_0\bar{x}'\geq yA_0x^*=wx^*$, we have that $\bar{x}'$ is an optimal solution to $\max\{wx\colon Ax\leq b'\}$.

Finally, we show part (ii). Suppose that $Ax\leq b$ and $Ax\leq b'$ each have integral solutions and let $\bar{z}$ be an optimal solution to $\max\{wx\colon Ax\leq b, x \text{ integral}\}$. By Theorem 1(ii), there exists an optimal solution x^* to $\max\{wx\colon Ax\leq b\}$ with $\|\bar{z}-x^*\|_\infty\leq n\Delta(A)$. Part (i), above, implies that there exists an optimal solution $\bar{x}'$ to $\max\{wx\colon Ax\leq b'\}$ with $\|x^*-\bar{x}'\|_\infty\leq n\Delta(A)\|b-b'\|_\infty$. Now, by Theorem 1(i), there exists an optimal solution $\bar{z}'$ to $\max\{wx\colon Ax \leq b', x \text{ integral}\}$ with $\|\bar{x}' - \bar{z}'\|_\infty \leq n\Delta(A)$. Putting this together, we have

$$\|\bar{z} - \bar{z}'\|_\infty \leq \|\bar{z} - x^*\|_\infty + \|x^* - \bar{x}'\|_\infty + \|\bar{x}' - \bar{z}'\|_\infty$$
$$\leq n\Delta(A)(\|b - b'\|_\infty + 2). \qquad \square \tag{10}$$

A result related to Theorem 1 was proven by Graver [16] (see also Blair and Jeroslow [3, Lemma 4.3]), who showed that for every integral matrix A there exists a finite 'test set' L of integral vectors such that: For any objective vector w and right-hand-side vector b and any nonoptimal feasible solution z to the integer program $\max\{wx\colon Ax \leq b, x \text{ integral}\}$ there exists a vector $l \in L$ such that $z + l$ is a feasible solution with greater objective value. The following result shows that if A is an $m \times n$ integral matrix, then we can take as a 'test set' simply the set of integral vectors with components at most $n\Delta(A)$ in absolute value. (An analysis of Blair and Jeroslow's proof [3, Lemma 4.3] will give a similar result.)

Theorem 6. *Let* $\max\{wx\colon Ax \leq b, x \text{ integral}\}$ *be an integer program with A an integral* $m \times n$ *matrix. Then, for each integral solution z of* $Ax \leq b$*, either z is an optimal solution to the integer program or there exists an integral solution* $\bar{z}$ *of* $Ax \leq b$ *with* $\|z - \bar{z}\|_\infty \leq n\Delta(A)$ *and* $w\bar{z} > wz$.

Proof. Let z be an integral solution of $Ax \leq b$. If z is not an optimal solution to $\max\{wx\colon Ax \leq b, x \text{ integral}\}$, then there exists an integral solution z^* of $Ax \leq b$ with $wz^* > wz$. Split A into submatrices A_1, A_2 such that $A_1 z^* \geq A_1 z$ and $A_2 z^* < A_2 z$. As in the proof of Theorem 1, we have $z^* - z = \lambda_1 g^1 + \cdots + \lambda_t g^t$ for some numbers $\lambda_i \geq 0$, $i = 1, \ldots, t$, and integral vectors $g^1, \ldots, g^t$ contained in the cone $C = \{x\colon A_1 x \geq 0, A_2 x \leq 0\}$ with $\|g^i\|_\infty \leq \Delta(A)$ for $i = 1, \ldots, t$, where t is the dimension of C.

If $\lambda_1 \geq 1$, then we have that $z + g^1 = z^* - (\lambda_1 - 1)g^1 - \lambda_2 g^2 - \cdots - \lambda_t g^t$ is an integral solution of $Ax \leq b$. So if $\lambda_1 \geq 1$ and $wg^1 > 0$, then we may set $\bar{z} = z + g^1$. Thus, we may assume that $wg^i \leq 0$ for all $i \in \{1, \ldots, t\}$ such that $\lambda_i \geq 1$. Let

$$\bar{z} = z^* - \lfloor\lambda_1\rfloor g^1 - \cdots - \lfloor\lambda_t\rfloor g^t = z + (\lambda_1 - \lfloor\lambda_1\rfloor)g^1 + \cdots + (\lambda_t - \lfloor\lambda_t\rfloor)g^t. \tag{11}$$

We have that $\bar{z}$ is an integral solution of $Ax \leq b$ with

$$\|z - \bar{z}\|_\infty \leq (\lambda_1 - \lfloor\lambda_1\rfloor)\|g^1\|_\infty + \cdots + (\lambda_t - \lfloor\lambda_t\rfloor)\|g^t\|_\infty \leq n\Delta(A). \tag{12}$$

Since $wg^i \leq 0$ for each $i \in \{1, \ldots, t\}$ such that $\lfloor\lambda_i\rfloor > 0$, we have $w\bar{z} \geq wz^*$. Thus $w\bar{z} > wz$. $\square$

For a linear system $Ax \leq b$, let $\{x\colon Ax \leq b\}_I$ denote the convex hull of its integral solutions. The following result of Wolsey [28, Theorem 2′], on the defining systems of $\{x\colon Ax \leq b\}_I$ as b varies, will be used in the next section. We use the proof technique of Blair and Jeroslow [3, Theorem 6.2], together with Theorem 6 to prove this result, as this allows us to bound the size of certain coefficients, as stated below.

Theorem 7. *For each integral $m \times n$ matrix A, there exists an integral matrix M, with entries at most $n^{2n}\Delta(A)^n$ in absolute value, such that for each vector b for which $Ax \leq b$ has an integral solution there exists a vector d_b such that $\{x\colon Ax \leq b\}_I = \{x\colon Mx \leq d_b\}$.*

Proof. We may assume that $A \neq 0$. Let K be the cone generated by the rows of A. If b is a vector such that $Ax \leq b$ has an integral solution, then, by linear programming duality, $\max\{wx\colon Ax \leq b, x \text{ integral}\}$ exists if and only if $w \in K$. Let $L = \{z \in Z^n\colon \|z\|_\infty \leq n\Delta(A)\}$ and for each $T \subseteq L$ let $G(T)$ be a finite set of integral vectors which generates the cone $C(T) = \{w\colon wz \leq 0 \text{ for all } z \in T\} \cap K$. Finally, let M be the matrix with rows $\bigcup\{G(T)\colon T \subseteq L\}$.

Suppose that b is a fixed vector such that $Ax \leq b$ has an integral solution. By replacing b by $\lfloor b \rfloor$ if necessary, we may assume that b is integral. Let d_b be the minimal vector (with respect to the $\leq$ ordering) such that $\{x\colon Ax \leq b\}_I \subseteq \{x\colon Mx \leq d_b\}$. (Since each row of M is in the cone K, each component of d_b is finite.) To prove that $\{x\colon Ax \leq b\}_I \supseteq \{x\colon Mx \leq d_b\}$, we will show that each valid inequality for $\{x\colon Ax \leq b\}_I$ is implied by $Mx \leq d_b$.

Let $\bar{w}x \leq t$ be an inequality such that $\bar{w} \neq 0$ and $\{x\colon Ax \leq b\}_I \subseteq \{x\colon \bar{w}x \leq t\}$. We may assume that $t = \max\{\bar{w}x\colon Ax \leq b, x \text{ integral}\}$. Let $\bar{z}$ be an integral solution of $Ax \leq b$ such that $\bar{w}\bar{z} = t$. Now let $\bar{T} = \{z \in L\colon A(\bar{z}+z) \leq b\}$. Since $\bar{z}$ is an optimal solution to $\max\{\bar{w}x\colon Ax \leq b, x \text{ integral}\}$, we have that $\bar{w} \in C(\bar{T})$. So there exist nonnegative numbers λ_g, for $g \in G(\bar{T})$, such that $\sum\{\lambda_g g\colon g \in G(\bar{T})\} = \bar{w}$. By Theorem 6, $\bar{z}$ is an optimal solution to $\max\{gx\colon Ax \leq b, x \text{ integral}\}$ for each $g \in G(\bar{T})$. Thus, letting $g(b) = \max\{gx\colon Ax \leq b, x \text{ integral}\}$ for each $g \in G(\bar{T})$, we have

$$\sum\{\lambda_g g(b)\colon g \in G(\bar{T})\} = \sum\{\lambda_g g\bar{z}\colon g \in G(\bar{T})\} = \bar{w}\bar{z} = t. \tag{13}$$

So $\bar{w}x \leq t$ is a nonnegative linear combination of inequalities in $Mx \leq d_b$, and, hence, is implied by $Mx \leq d_b$.

To bound the size of the entries in M, notice that Cramer's rule implies that there exists an integral matrix D such that $K = \{x\colon Dx \leq 0\}$ and each entry of D having absolute value at most $\Delta(A)$. Thus, for each $T \subseteq L$ the cone $C(T)$ is defined by a linear system $Fx \leq 0$ for some integral matrix F with entries at most $n\Delta(A)$ in absolute value. Using Cramer's rule, it follows that for each $T \subseteq L$ the cone $C(T)$ can be generated by a finite set of integral vectors, each of which has components which are at most $n!(n\Delta(A))^n \leq n^{2n}\Delta(A)^n$ in absolute value. □

For an integral matrix A, the bound given above on the size of the entries in M is polynomial in n and the size of the largest entry in A, and is independent of the number of rows of A. Thus, this bound implies the result, of Karp and Papadimitriou [21], that if a polyhedron $P \subseteq Q^n$ can be defined by a system of integral linear inequalities, each of which has size at most σ, then the convex hull of the integral points in P can be defined by a system of linear inequalities, each of which has size bounded above by a polynomial function of n and σ.

Remarks. (1) Blair and Jeroslow [1, 2] proved their results in the more general context of mixed integer programming. It is straightforward, however, to modify the proof of Theorem 1 to obtain the following mixed integer programming result:

Let A be an integral $m\times k$ matrix, B an integral $m\times(n-k)$ matrix, and w, v, and b vectors such that $\max\{wx+vy\colon Ax+By\leq b, x \text{ integral}\}$ has an optimal solution. Then, for each optimal solution (x^1, y^1) to $\max\{wx+vy\colon Ax+By\leq b\}$, there exists an optimal solution (x^2, y^2) to $\max\{wx+vy\colon Ax+By\leq b, x \text{ integral}\}$ such that $\|(x^1, y^1)-(x^2, y^2)\|_\infty\leq n\Delta((A|B))$. And, for each optimal solution (x^3, y^3) to $\max\{wx+vy\colon Ax+By\leq b, x \text{ integral}\}$, there exists an optimal solution (x^4, y^4) to $\max\{wx+vy\colon Ax+By\leq b\}$ such that $\|(x^3, y^3)-(x^4, y^4)\|_\infty\leq n\Delta((A|B))$.

Using this, one can prove the mixed integer programming analogues of Corollary 2, Corollary 3, and Theorem 5(ii).

In a similar way, one can obtain an analogue of Theorem 6. Note, however, that the mixed integer programming version of Theorem 6 does not imply a finite 'test set' result for mixed integer programs. This is due to the fact that, unlike the integer programming case, there may exist infinitely many vectors in $\{(x, y)\colon x \text{ integral}, \|(x, y)\|_\infty\leq n\Delta(A|B)\}$. Thus one cannot use the proof of Theorem 7 to obtain an analogous result for mixed integer programming. (In fact, the mixed integer programming analogue of Theorem 7 is not true, since, for example, the convex hull of $\{(x_1, x_2)\colon x_1\geq 0, x_2\geq 0, x_1+x_2\leq 1, x_2\leq\alpha, x_1 \text{ integral}\}$, for any α such that $0<\alpha<1$, has $x_1+(1/\alpha)x_2\leq 1$ as a facet-inducing inequality.) Similarly, the mixed integer programming version of Theorem 1 does not imply a mixed integer analogue of Corollary 4.

However, observe that if A, B and b are integral and $\max\{wx+vy\colon Ax+By\leq b, x \text{ integral}\}$ has an optimal solution then it has one such that the denominator of each component of the vector y is the determinant of a square submatrix of B. This observation can be used to prove a mixed integer analogue of Corollary 4 and a finite 'test set' consequence of Theorem 6 for the case where b is integral. This latter finite 'test set' result can be used to modify the proof of Theorem 7 to obtain the following:

Let A be an integral $m\times k$ matrix and let B be an integral $m\times(n-k)$ matrix. Then there exist integral matrices M and N, each with entries at most $n^{2n}\Delta(B)^{2n}\Delta((A|B))^n$ in absolute value, such that for each integral vector b for which $\{(x, y)\colon Ax+By\leq b, x \text{ integral}\}$ is nonempty, there exists a vector d_b such that $\{(x, y)\colon Mx+Ny\leq d_b\}$ is equal to the convex hull of $\{(x, y)\colon Ax+By\leq b, x \text{ integral}\}$.

(2) The proof of Theorem 5(i) remains valid, even when A is an irrational matrix, if we replace $\Delta(A)$ by $\Delta'(A)$, the maximum, over all invertible submatrices B of A, of the absolute values of the entries of B^{-1}.

3. The Chvátal rank of matrices

If all vectors in a polyhedron P satisfy the linear inequality $ax \leqslant \beta$, where a is an integral vector and β is a number, then each integral vector in P satisfies the *Chvátal cut* $ax \leqslant \lfloor \beta \rfloor$. Denote by P' the set of vectors which satisfy every Chvátal cut of P. It is easy to see that $P_I \subseteq P'$, where P_I denotes the convex hull of integral vectors in P. Schrijver [24], continuing the work of Chvátal [5] and Gomory [13], proved that P' is a polyhedron and that $P^{(t)} = P_I$ for some natural number t, where $P^{(0)} = P$ and $P^{(i)} = P^{(i-1)'}$ for all $i \geqslant 1$. The least number t such that $P^{(t)} = P_I$ is the *Chvátal rank* of P. Results on the Chvátal rank of combinatorially described polyhedra can be found in Boyd and Pulleyblank [4] and Chvátal [5, 6, 7].

We define the *Chvátal rank of matrix* A to be the supremum over all integral vectors b of the Chvátal rank of $\{x\colon Ax \leqslant b\}$. It is convenient also to define the *strong Chvátal rank* of A, which is the Chvátal rank of the matrix

$$\begin{bmatrix} I \\ -I \\ A \\ -A \end{bmatrix}.$$

Then the well-known theorem of Hoffman and Kruskal [20] is that an integral matrix has strong Chvátal rank 0 if and only if it is totally unimodular. Moreover, Hoffman and Kruskal showed that an integral matrix A has Chvátal rank 0 if and only if A^T is unimodular. (A matrix C is called *unimodular* if for each submatrix B consisting of r ($:=$ rank C) linearly independent columns of C, the g.c.d. of the subdeterminants of B of order r is 1.) In Edmonds and Johnson [9] and Gerards and Schrijver [11], characterisations of classes of matrices having Chvátal rank 1 are given. The main result of this section is that every matrix has a finite Chvátal rank.

To prove this result we need the following theorem of Cook, Coullard, and Turán [8]. For completeness, a proof of this result is given below. In the proof, we use a lemma of Schrijver [24] which implies that if F is a face of a polyhedron P then $P^{(k)} \cap F \subseteq F^{(k)}$ for all $k \geqslant 0$. We also use the fact that if T is an affine transformation from Q^n to Q^n which maps Z^n onto Z^n, then for any polyhedron $P \subseteq Q^n$ we have $T(P^{(k)}) = T(P)^{(k)}$ for all $k \geqslant 0$.

Theorem 9. *If $P \subseteq Q^n$ is a polyhedron with $P \cap Z^n = \emptyset$, then $P^{(n^{2n}2^{n^3})} = \emptyset$.*

Proof. The proof is by induction on n. It is easy to see that the result is true for $n = 1$. Suppose $n \geqslant 2$ and that for all polyhedra $G \subseteq Q^{n-1}$ with $G \cap Z^{n-1} = \emptyset$, we have $G^{(\gamma_{n-1})} = \emptyset$, where $\gamma_{n-1} = (n-1)^{2(n-1)}2^{(n-1)^3}$.

Let $P \subseteq Q^n$ be a polyhedron with $P \cap Z^n = \emptyset$. It follows from a result of Grötschel, Lovász, and Schrijver [18] that there exists a nonzero integral vector w such that $|w(x - x')| < n(n+1)2^{(1/2)n^2}$ for all x, $x' \in P$. (In the case where P is a bounded polyhedron, the existence of such a vector w was shown by Lenstra [22], see also

Grötschel, Lovász, and Schrijver [17].) Let

$$\gamma_n = n^{2n}2^{n^3} \geq n(n+1)2^{(1/2)n^2}(\gamma_{n-1}+1)+1. \tag{14}$$

We will complete the proof by showing that $P^{(\gamma_n)} = \emptyset$.

We may assume that the components of w are relatively prime integers. Let $\beta = \lfloor \max\{wx: x \in P\} \rfloor$. It follows that $P^{(1)} \subseteq \{x: wx \leq \beta\}$. If $P^{(1)} = \emptyset$ we are finished, so suppose not. Let $G = P^{(1)} \cap \{x: wx = \beta\}$. Since $\{x: wx = \beta\}$ contains integral vectors, there exists an affine transformation T which maps Z^n onto Z^n and $\{x: wx = \beta\}$ onto $\{x: x_n = 0\}$. Let $\bar{G} = \{(x_1, \ldots, x_{n-1}) \in Q^{n-1}: (x_1, \ldots, x_{n-1}, 0) \in T(G)\}$. By assumption, $\bar{G}^{(\gamma_{n-1})} = \emptyset$. But this implies that $T(G)^{(\gamma_{n-1})} = \emptyset$ and hence that $G^{(\gamma_{n-1})} = \emptyset$. Thus, by the lemma of Schrijver [24] mentioned above, we have $P^{(\gamma_{n+1}+1)} \cap G = \emptyset$. So $P^{(\gamma_{n-1}+2)} \subseteq \{x: wx \leq \beta - 1\}$. Since $P \subseteq \{x: wx > \beta - n(n+1)2^{(1/2)n^2}\}$, repeating this procedure at most $n(n+1)2^{(1/2)n^2} - 1$ times, we obtain the empty set. Hence $P^{(\gamma_n)} = \emptyset$. □

We will also make use of the following consequence of this theorem.

Corollary 9. *Let $P \subseteq Q^n$ be a polyhedron such that $P \cap Z^n \neq \emptyset$ and let w be an integral vector such that $q = \max\{wx: x \in P_I\}$ exists. Then $P^{(r)} \subseteq \{x: wx \leq q\}$, where $r = (n^{2n}2^{n^3}+1)(\lfloor \max\{wx: x \in P\} \rfloor - q) + 1$.*

Proof. Let $\beta = \lfloor \max\{wx: x \in P\} \rfloor$, which exists since q exists. We have $P^{(1)} \subseteq \{x: wx \leq \beta\}$. If $q = \beta$ we are finished, so suppose $q < \beta$. Let $G = P^{(1)} \cap \{x: wx = \beta\}$. Since $G \cap Z^n = \emptyset$, we have $G^{(n^{2n}2^{n^3})} = \emptyset$, by Theorem 8. Using the lemma of Schrijver [24], this implies that $P^{(n^{2n}2^{n^3}+1)} \cap \{x: wx = \beta\} = \emptyset$. So $P^{(n^{2n}2^{n^3}+2)} \subseteq \{x: wx \leq \beta - 1\}$. Repeating this operation $(\beta - q - 1)$ times, we have $P^{(r)} \subseteq \{x: wx \leq q\}$. □

We are now ready to prove our finite Chvátal rank theorem.

Theorem 10. *The Chvátal rank of an integral $m \times n$ matrix A is at most $2^{n^3+1}n^{5n}\Delta(A)^{n+1}$.*

Proof. Let M be an integral matrix satisfying the conditions given in Theorem 7, for the matrix A. We have $\max\{\|m\|_1: m$ is a row of $M\} \leq n^{2n+1}\Delta(A)^n$. Let

$$k = (n^{2n}2^{n^3}+1)(n^{2n+2}\Delta(A)^{n+1})+1 \tag{15}$$

and let $P = \{x: Ax \leq b\}$ for some vector b. Since $k \leq 2^{n^3+1}n^{5n}\Delta(A)^{n+1}$, it suffices to show that $P^{(k)} = P_I$. If $P \cap Z^n = \emptyset$, then, by Theorem 8, $P^{(k)} = \emptyset = P_I$. Suppose $P \cap Z^n \neq \emptyset$. We have $P_I = \{x: Mx \leq d_b\}$ for some vector d_b. Let m be a row of M and let $q = \max\{mx: x \in P_I\}$. By Corollary 2,

$$\max\{mx: x \in P\} - q \leq n\Delta(A)\|m\|_1 \leq n^{2n+2}\Delta(A)^{n+1}. \tag{16}$$

Thus, by Corollary 9, $P^{(k)} \subseteq \{x: mx \leq q\}$. So $P^{(k)} \subseteq \{x: Mx \leq d_b\}$, which implies that $P^{(k)} = P_I$. □

Remarks. (1) A consequence of Theorem 10 is that the number of Chvátal cuts that must be added to a linear system $Ax \leq b$ to obtain a defining system for $\{x\colon Ax \leq b\}_I$ can be bounded above by a function of the matrix A, independent of the right-hand-side vector b. Indeed, for each linear system $Ax \leq b$ with A an integral $m \times n$ matrix of Chvátal rank t, the polyhedron $\{x\colon Ax \leq b\}_I$ is the solution set of a system of inequalities

$$\begin{aligned} &Ax \leq b, \\ &\alpha_i x \leq \beta_i \quad (i = 1, \dots, N) \end{aligned} \tag{17}$$

where $N \leq 2^n t n^{tn} S^n$ (with S denoting the maximum of the absolute values of the entries of A) and for each $k = 1, \dots, N$, the inequality $\alpha_k x \leq \beta_k$ is a Chvátal cut for the polyhedron $\{x\colon Ax \leq b, \alpha_i x \leq \beta_i, i = 1, \dots, k-1\}$. To prove this, we use the result of Schrijver [24] that if $P = \{x\colon Mx \leq d\}$, where M is integral and $Mx \leq d$ is a totally dual integral system, then $P' = \{x\colon Mx \leq \lfloor d \rfloor\}$. (A linear system $Mx \leq d$ is a *totally dual integral system* if $\min\{yd\colon yM = w, y \geq 0\}$ can be achieved by an integral vector for each integral w for which the minimum exists.) Giles and Pulleyblank [15] proved that every polyhedron can be defined by such a totally dual integral system (see also Schrijver [25]). Their proof, together with Carathéodory's theorem, implies that if $Dx \leq f$ is a linear system with D an integral $m \times n$ matrix, each entry of which has absolute value at most T, then there exists a totally dual integral system $D'x \leq f'$ with $\{x\colon D'x \leq f'\} = \{x\colon Dx \leq f\}$ such that each entry of D' is an integer with absolute value at most nT. This implies that if $P = \{x\colon Ax \leq b\}$, then for each $i = 1, \dots, t$ the polyhedron $P^{(i)}$ can be defined by a linear system $M^i x \leq b^i$ where each inequality is a Chvátal cut for $P^{(i-1)}$ and where M^i is an integral matrix having at most $(2n^i S)^n$ rows. (This technique for bounding the size of M^i is also used in Boyd and Pulleyblank [4].)

(2) Professor C. Blair pointed out to us several relations of his and Jeroslow's work with our results. Especially, the fact that each matrix has a finite Chvátal rank can be seen to be equivalent to their result that 'each integer programming value function is a Gomory function'. Here we shall discuss this relation.

A function $f\colon \mathbb{Q}^m \to \mathbb{Q}$ is a *Gomory function* if there exist rational matrices $M_1, \dots, M_t$ so that $M_1, \dots, M_{t-1}$ are nonnegative, and so that, for each $b \in \mathbb{Q}^m$,

$$f(b) = \max_j (M_1 \lceil M_2 \lceil \cdots \lceil M_t b \rceil \cdots \rceil \rceil)_j \tag{18}$$

(here $\lceil\,,\rceil$ denotes component-wise upper integer parts; M_t has m columns, and M_i has the same number of rows as M_{i-1} has columns $(i = 2, \dots, t)$; the maximum ranges over all coordinates j of the vector $(M_i \lceil M_2 \lceil \cdots \lceil M_t b \rceil \cdots \rceil \rceil)$.)

Blair and Jeroslow [3, Theorems 5.1 and 5.2] showed:

Blair–Jeroslow Theorem. *For each rational $m \times n$-matrix A and row vector $c \in \mathbb{Q}^n$ with* $\min\{cx\colon Ax = 0, x \geq 0\}$ *finite, there exist Gomory functions f, $g\colon \mathbb{Q}^m \to \mathbb{Q}$ so that, for* *each* $b \in \mathbb{Q}^m$,

$$\text{(i)}\quad f(b) \leqslant 0 \text{ if and only if } \{x \mid x \geqslant 0;\ Ax = b;\ x \text{ integral}\} \text{ is nonempty;}$$
$$\text{(ii)}\quad g(b) = \min\{cx \mid x \geqslant 0;\ Ax = b;\ x \text{ integral}\} \quad \text{iff } f(b) \leqslant 0. \tag{19}$$

Proposition. *The Blair–Jeroslow theorem is equivalent to each rational matrix having finite Chvátal rank.*

Proof (sketch). I. We first show that the Blair–Jeroslow theorem implies that each rational matrix has a finite Chvátal rank. One easily checks that it suffices to derive from the Blair–Jeroslow theorem that for each rational matrix A and vector b, the Chvátal rank of the polyhedron $\{x \mid x \geqslant 0;\ Ax = b\}$ has an upper bound only depending on A.

Choose a rational matrix A. By Theorem 7, there exists a matrix C so that for each vector b there exists a vector d_b with $\{x \mid x \geqslant 0;\ Ax = b\}_I = \{x \mid x \geqslant 0;\ Cx \leqslant d_b\}$.

Take any row c of C. By the Blair–Jeroslow theorem there exist Gomory functions f and g with the properties described in (19). Hence there exist matrices $M_1, \ldots, M_t$ and $N_1, \ldots, N_{t'}$, so that $M_1, \ldots, M_{t-1}, N_1, \ldots, N_{t'-1} \geqslant 0$ and, for each $b \in \mathbb{Q}^m$,

$$\min\{cx \mid x \geqslant 0;\ Ax = b;\ x \text{ integral}\} = \max\{uM_1\lceil M_2 \cdots \lceil M_t b\rceil \cdots \rceil + vN_1\lceil N_2 \cdots \lceil N_{t'} b\rceil \cdots \rceil \mid u \geqslant 0;\ v \geqslant 0;\ v\mathbf{1} = 1\}. \tag{20}$$

Without loss of generality, $t = t'$ (otherwise add some matrices I). By taking for b any column of A one sees that for each $u \geqslant 0$, $v \geqslant 0$ with $v\mathbf{1} = 1$ one has:

$$uM_1\lceil M_2 \cdots \lceil M_t A\rceil \cdots \rceil + vN_1\lceil N_2 \cdots \lceil N_t A\rceil \cdots \rceil \leqslant c. \tag{21}$$

Hence, for each b,

$$\min\{cx \mid x \geqslant 0;\ Ax = b;\ x \text{ integral}\} \leqslant \max\{w\lceil P_2 \cdots \lceil P_t b\rceil \cdots \rceil \mid w \geqslant 0;\ w\lceil P_2 \cdots \lceil P_t A\rceil \cdots \rceil \leqslant c\}, \tag{22}$$

where

$$P_i := \begin{bmatrix} M_i & 0 \\ 0 & N_i \end{bmatrix} \quad (i = 2, \ldots, t-1), \qquad P_t := \begin{bmatrix} M_t \\ N_t \end{bmatrix} \tag{23}$$

((22) follows by taking $w := (uM_1 vN_1)$).

By applying LP-duality to the maximum in (22) we see that for each b:

$$\min\{cx \mid x \geqslant 0;\ Ax = b;\ x \text{ integral}\} \leqslant \min\{cx \mid x \geqslant 0;\ \lceil P_2 \cdots \lceil P_t A\rceil \cdots \rceil x \leqslant \lceil P_2 \cdots \lceil P_t b\rceil \cdots \rceil\}. \tag{24}$$

Hence equality follows.

Let $t_{\max}$ be the maximum of the t above when c runs over all rows of C. Then we know that the Chvátal rank of $\{x \mid x \geqslant 0;\ Ax = b\}$ is at most $t_{\max}$, for each $b \in \mathbb{Q}^m$.

II. We next show the reverse implication. Choose matrix A and vector c. As, by assumption, the matrix

$$\begin{bmatrix} A \\ -A \\ I \end{bmatrix}$$

has finite Chvátal rank, we know that there exist matrices $P_1, \ldots, P_t$ so that $P_1, \ldots, P_{t-1} \geq 0$ and so that, for each $b \in \mathbb{Q}^m$,

$$\{x \mid x \geq 0;\ Ax = b\}_I = \{x \mid x \geq 0;\ \lceil P_1 \lceil \cdots \lceil P_t A \rceil \cdots \rceil x \geq \lceil P_1 \lceil \cdots \lceil P_t b \rceil \cdots \rceil\}. \tag{25}$$

Hence, with LP-duality,

$$\begin{aligned} &\min\{cx \mid x \geq 0;\ Ax = b;\ x \text{ integral}\} \\ &\quad = \min\{cx \mid x \geq 0;\ \lceil P_1 \cdots \lceil P_t A \rceil \cdots \rceil x \geq \lceil P_1 \cdots \lceil P_t b \rceil \cdots \rceil\} \\ &\quad = \max\{y \lceil P_1 \cdots \lceil P_t b \rceil \cdots \rceil \mid y \geq 0;\ y \lceil P_1 \cdots \lceil P_t A \rceil \cdots \rceil \leq c\}. \end{aligned} \tag{26}$$

Let the rows of the matrix M, say, be the vertices of the polyhedron $\{y \geq 0 \mid y \lceil P_1 \cdots \lceil P_t A \rceil \cdots \rceil \leq c\}$, and let the rows of the matrix N, say, be the extremal (infinite) rays of this polyhedron. Then the maximum in (26) is equal to:

$$\max\{uN \lceil P_1 \cdots \lceil P_t b \rceil \cdots \rceil + vM \lceil P_1 \cdots \lceil P_t b \rceil \cdots \rceil \mid u \geq 0;\ v \geq 0;\ v\mathbf{1} = 1\}. \tag{27}$$

Hence defining, for $b \in \mathbb{Q}^m$,

$$\begin{aligned} f(b) &:= \max_j (N \lceil P_1 \cdots \lceil P_t b \rceil \cdots \rceil)_j, \\ g(b) &:= \max_j (M \lceil P_1 \cdots \lceil P_t b \rceil \cdots \rceil)_j, \end{aligned} \tag{28}$$

gives Gomory functions satisfying (19). □

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